

Time-inhomogeneous fractional Poisson processes defined by the multistable subordinator

Luisa Beghin ^{*} Costantino Ricciuti [†]

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Abstract

The space-fractional and the time-fractional Poisson processes are two well-known models of fractional evolution. They can be constructed as standard Poisson processes with the time variable replaced by a stable subordinator and its inverse, respectively. The aim of this paper is to study non-homogeneous versions of such models, which can be defined by means of the so-called multistable subordinator (a jump process with non-stationary increments), denoted by $H := H(t), t \geq 0$. Firstly, we consider the Poisson process time-changed by H and we obtain its explicit distribution and governing equation. Then, by using the right-continuous inverse of H , we define an inhomogeneous analogue of the time-fractional Poisson process.

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1 Introduction

Non-homogeneous subordinators are univariate additive processes with non-decreasing sample paths. Such processes, together with their right continuous inverses, have recently been studied in [17], where they are also used as random clock for time-changed processes. Recall that an additive process is characterized by independent increments and is stochastically continuous, null at the origin and with cadlag trajectories (for a deeper insight consult [21]). If, in addition, we assume stationarity of the increments, additive processes reduce to the standard Lévy ones.

A non-homogeneous subordinator (without drift) is completely characterized by a set $\{\nu_t, t \geq 0\}$ of Lévy measures on \mathbb{R}^+ , such that

$$\nu_t(0) = 0 \quad \int_0^\infty (x \wedge 1) \nu_t(dx) < \infty, \quad t \geq 0.$$

^{*}luisa.beghin@uniroma1.it, Department of Statistical Sciences, Sapienza - University of Rome

[†]costantino.ricciuti@uniroma1.it (corresponding author), Department of Statistical Sciences, Sapienza-University of Rome.

If $\nu_t(\mathbb{R}^+) < \infty$, for any $t \geq 0$, then the process reduces to an inhomogeneous Compound Poisson Process (hereafter CPP), while condition $\nu_t(\mathbb{R}^+) = \infty$, for any $t \geq 0$, ensures that the process is strictly increasing almost surely. Under suitable conditions (see [17]), the Laplace transform of the increments of an inhomogeneous subordinator T has the form

$$\mathbb{E}e^{-u(T(t)-T(s))} = e^{-\int_s^t f(u,\tau)d\tau} \quad 0 \leq s \leq t, \quad (1.1)$$

where $u \rightarrow f(u, t)$ is a Bernstein function for each $t \geq 0$, having the following form

$$f(u, t) = \int_0^\infty (1 - e^{-ux})\nu_t(dx). \quad (1.2)$$

Among inhomogeneous subordinators, we are particularly interested in the so-called multistable subordinators (see [15], [17], [9]). These processes extend the well-known stable subordinators by letting the stability index α evolve autonomously in time: for this reason they have been proved to be particularly useful in modelling phenomena, both in finance and in natural sciences, where the intensity of the jumps is itself time-dependent. The multistable subordinator is fully characterized by a Lévy measure of the form

$$\nu_t(dx) = \frac{\alpha(t)x^{-\alpha(t)-1}}{\Gamma(1-\alpha(t))}dx, \quad x > 0,$$

where $t \rightarrow \alpha(t)$ has values in $(0, 1)$. Throughout the paper we will denote a multistable subordinator by $H := \{H(t), t \geq 0\}$. It is known (see [17]) that, for each $t \geq 0$, the random variable $H(t)$ is absolutely continuous and its density solves

$$\frac{\partial}{\partial t}q(x, t) = -\frac{\partial^{\alpha(t)}}{\partial x^{\alpha(t)}}q(x, t), \quad q(x, 0) = \delta(x),$$

where $\frac{\partial^{\alpha(t)}}{\partial x^{\alpha(t)}}$ is the Riemann-Liouville derivative with time-varying order.

Since, in this case, $f(u, t) = u^{\alpha(t)}$, the increment from s to t has Laplace transform

$$\mathbb{E}e^{-u(H(t)-H(s))} = e^{-\int_s^t u^{\alpha(\tau)}d\tau}, \quad 0 \leq s \leq t.$$

The first part of the present paper has been inspired by [16], [20] and [18]. In particular, in [16] the authors study the composition of a Poisson process with a stable subordinator. The resulting process, called space-fractional Poisson process, is also a subordinator, namely a point process with upward jumps, with arbitrary, integer size.

Let now $N := \{N(t), t \geq 0\}$ be a homogeneous Poisson process with intensity $\lambda > 0$ and let H be a multistable subordinator independent of N . We consider here the point process $X := \{X(t), t \geq 0\}$, where, for any $t \geq 0$, $X(t) := N(H(t))$, with positive integer values, that we call Space-Multifractional Poisson Process (hereafter SMPP). We prove that its state probabilities $p_k(t) = \Pr\{X(t) = k\}$ satisfy the following system of difference-differential equations:

$$\begin{cases} \frac{d}{dt}p_k(t) = -\lambda^{\alpha(t)}(I - B)^{\alpha(t)}p_k(t), & k = 0, 1, 2, \dots \\ p_k(0) = \delta_{k,0}, \end{cases} \quad (1.3)$$

where B is the shift operator such that $Bp_k(t) = p_{k-1}(t)$, and $\delta_{k,0}$ denotes the Kronecker delta function. The first equation in (1.3) is a time-inhomogeneous extension of the Space-Fractional Poisson governing equation studied in [16] (see also [2] for the compound case). This result confirms the validity of the time-inhomogeneous version of the Phillips' formula, which was proved in [17] for self-adjoint Markov generators only, and therefore it could not be taken for granted in the case of Poisson generators. In other words, referring to the general theory of Markov processes (see, for example, [8]), we say that the evolution of X is governed by a propagator (or two parameter semigroup) with time-dependent adjoint generator given by $\lambda^{\alpha(t)}(I - B)^{\alpha(t)}$.

In the second part of the present paper we study the so called Time-Multifractional Poisson Process (hereafter TMPP). It is obtained by time-changing the standard Poisson process via the right continuous inverse of a multistable subordinator, which is defined as

$$L(x) = \inf\{t \geq 0 : H(t) > x\}.$$

We recall that the classical time-fractional Poisson process is a renewal process with i.i.d Mittag-Leffler waiting times, having a deep connection to fractional calculus. It has been introduced and studied by [12], [3], [4], [19], [6], [5] and many others. In [13] the authors show that it can be constructed by time-changing a Poisson process via an independent inverse stable subordinator.

The idea of time-changing Markov processes via non-homogeneous subordinators has been developed in [17]. Moreover, in [15] the TMPP arises as a scaling limit of a continuous time random walk, but its distributional properties are not investigated there. We prove here that non-homogeneity has an impact on the distribution of the waiting times, which are independent but no longer identically distributed.

Very recently, some authors ([10] and [11]) considered some extensions of the time-fractional Poisson process, which are inhomogeneous in a different sense from ours. The difference consists in the fact that they analyse the time-change of an inhomogeneous Poisson process by the inverse of a homogeneous stable subordinator.

2 Preliminary results

In view of what follows, we preliminarily need the following extension of Theorem 30.1, p.197 in [21], to the case of non-homogeneous subordinators.

Proposition 2.1 *Let $M := \{M(t), t > 0\}$ be a Lévy subordinator such that $\mathbb{E}e^{-uM(t)} = e^{-tg(u)}$ and let $T := \{T(t), t > 0\}$ be a non-homogeneous subordinator (without drift) with Lévy measure ν_t and Bernstein function $f(\cdot, \cdot)$ as defined in (1.1) and (1.2). Let $Z := \{Z(t) = M(T(t)), t > 0\}$ be the time changed process. Then*

- i) Z is a non-homogeneous subordinator (without drift)*
- ii) Z has time-dependent Lévy measure*

$$\nu_t^*(dx) = \int_0^\infty \Pr(M(s) \in dx) \nu_t(ds) \quad (2.1)$$

Proof. *i)* The fact that Z is non-decreasing is obvious, since Z is given by the composition of non decreasing processes. It remains to prove independence of increments and stochastic continuity. First we prove that Z has independent increments. By Kac's theorem on characteristic functions (see [1], p.18), it is sufficient to prove that, for any $0 \leq t_1 \leq t_2 \leq t_3$,

$$\mathbb{E}e^{iy_1(Z(t_3)-Z(t_2))+iy_2(Z(t_2)-Z(t_1))} = \mathbb{E}e^{iy_1(Z(t_3)-Z(t_2))}\mathbb{E}e^{iy_2(Z(t_2)-Z(t_1))} \quad \forall (y_1, y_2) \in \mathbb{R}^2.$$

For the sake of simplicity, we use the notation $T(t_j) = T_j$. A simple conditioning argument yields

$$\begin{aligned} \mathbb{E}e^{iy_1(Z(t_3)-Z(t_2))+iy_2(Z(t_2)-Z(t_1))} &= \mathbb{E}[\mathbb{E}(e^{iy_1(M(T_3)-M(T_2))+iy_2(M(T_2)-M(T_1))}|T_1, T_2, T_3)] \\ &= \mathbb{E}[\mathbb{E}(e^{iy_1(M(T_3)-M(T_2))}|T_2, T_3)\mathbb{E}(e^{iy_2(M(T_2)-M(T_1))}|T_1, T_2)], \end{aligned}$$

where the last step follows by the fact that M has independent increments. Now, since M has stationary increments, we have

$$\mathbb{E}[\mathbb{E}(e^{iy_1(M(T_3)-M(T_2))}|T_2, T_3)\mathbb{E}(e^{iy_2(M(T_2)-M(T_1))}|T_1, T_2)] = \mathbb{E}e^{iy_1M(T_3-T_2)}\mathbb{E}e^{iy_2M(T_2-T_1)}, \quad (2.2)$$

where, in the last equality, we have taken into account that T has independent increments and thus $M(T_3 - T_2)$ and $M(T_2 - T_1)$ are stochastically independent. By using again the same conditioning argument, it is now immediate to observe that the right hand side of (2.2) can be written as

$$\mathbb{E}e^{iy_1(M(T_3)-M(T_2))}\mathbb{E}e^{iy_2(M(T_2)-M(T_1))},$$

since M has stationary increments, and this concludes the proof of the independence of increments of Z .

We now recall that a process $Y(t)$ is said to be stochastically continuous at time t if $P(|Y(t+h) - Y(t)| > a) \rightarrow 0$, as $h \rightarrow 0$, for any $a > 0$. Then, denoting by $\mu_{t,t+h}$ the law of $T(t+h) - T(t)$ and using the stationarity of the increments of M , we have that

$$\begin{aligned} \Pr\{|Z(t+h) - Z(t)| > a\} &= \Pr\{|M(T(t+h)) - M(T(t))| > a\} \\ &= \int_0^\infty \Pr\{|M(u)| > a\} \mu_{t,t+h}(du) \\ &= \int_0^\delta \Pr\{|M(u)| > a\} \mu_{t,t+h}(du) + \int_\delta^\infty \Pr\{|M(u)| > a\} \mu_{t,t+h}(du) \\ &\leq \sup_{u \in (0, \delta)} \Pr\{|M(u)| > a\} + \Pr\{|T(t+h) - T(t)| > \delta\}, \end{aligned}$$

where $\delta > 0$ can be arbitrarily small. Now, by letting δ and h go to zero, stochastic continuity of M and T produces the desired result.

ii) By using a simple conditioning argument, we have that

$$\begin{aligned} \mathbb{E}e^{-uM(T(t))} &= \int_0^\infty \mathbb{E}e^{-uM(s)} \Pr\{T(t) \in ds\} \\ &= \int_0^\infty e^{-sg(u)} \Pr\{T(t) \in ds\} \end{aligned}$$

$$= e^{-\int_0^t f(g(u), \tau) d\tau}.$$

Thus the Bernstein function of $M(T(t))$ has the form

$$\begin{aligned} f(g(u), \tau) &= \int_0^\infty (1 - e^{-g(u)z}) \nu_\tau(dz) \\ &= \int_0^\infty (1 - \mathbb{E} e^{-uM(z)}) \nu_\tau(dz) \\ &= \int_0^\infty \nu_\tau(dz) \int_0^\infty (1 - e^{-ux}) \Pr\{M(z) \in dx\} \\ &= \int_0^\infty (1 - e^{-ux}) \int_0^\infty \Pr\{M(z) \in dx\} \nu_\tau(dz) \\ &= \int_0^\infty (1 - e^{-ux}) \nu_\tau^*(dx) \end{aligned}$$

and the proof is complete. ■

3 Space-Multifractional Poisson process

Consider a standard Poisson process N , with rate $\lambda > 0$, and a multistable subordinator H with index $\alpha(t)$. We define the SMPP as the time-changed process $\{N(H(t)), t \geq 0\}$. Such a process is completely characterized by its time-dependent Lévy measure and by its transition probabilities, which are given in the following theorem.

Theorem 3.1 *The SMPP $X(t) := N(H(t))$, for any $t \geq 0$,
i) is a non-homogeneous subordinator and has Lévy measure*

$$\nu_t^*(dx) = \lambda^{\alpha(t)} \sum_{n=1}^{\infty} (-1)^{n+1} \binom{\alpha(t)}{n} \delta_n(dx), \quad (3.1)$$

ii) has the following transition probabilities

$$\Pr\{X(\tau+t) = k+n | X(\tau) = k\} = \begin{cases} \sum_{r=1}^{\infty} \frac{(-1)^{n+r}}{r!} \int_{[\tau, \tau+t]^r} \lambda^{\beta_r(s)} \binom{\beta_r(s)}{n} ds_1 \dots ds_r, & n \geq 1 \\ e^{-\int_{\tau}^{\tau+t} \lambda^{\alpha(s)} ds} & n = 0, \end{cases} \quad (3.2)$$

where

$$\beta_r(s) := \beta_r(s_1, \dots, s_r) = \sum_{j=1}^r \alpha(s_j).$$

Proof. i) The fact that X is a non-homogeneous subordinator is a consequence of Prop. 2.1. Denoting respectively by $\nu_t(dx)$ and $\nu_t^*(dx)$ the Lévy measures of H and X , we apply (2.1) and obtain

$$\nu_t^*(dx) = \int_0^\infty \Pr(N(s) \in dx) \nu_t(ds)$$

$$\begin{aligned}
&= \int_0^\infty \sum_{k=1}^\infty e^{-\lambda s} \frac{(\lambda s)^k}{k!} \delta_k(dx) \frac{\alpha(t) s^{-\alpha(t)-1}}{\Gamma(1-\alpha(t))} ds \\
&= \sum_{k=1}^\infty \frac{\alpha(t) \lambda^{\alpha(t)} \Gamma(k-\alpha(t))}{\Gamma(1-\alpha(t)) k!} \delta_k(dx) \\
&= \sum_{k=1}^\infty \frac{\alpha(t) \lambda^{\alpha(t)} (k-\alpha(t)-1)(k-\alpha(t)-2)\dots(-\alpha(t)) \Gamma(-\alpha(t))}{k! (-\alpha(t)) \Gamma(-\alpha(t))} \delta_k(dx) \\
&= \lambda^{\alpha(t)} \sum_{k=1}^\infty (-1)^{k+1} \binom{\alpha(t)}{k} \delta_k(dx).
\end{aligned}$$

ii) The probability generating function of the increment $X(\tau+t) - X(\tau)$ has the following form

$$\begin{aligned}
G(u, \tau, t) &= \mathbb{E} u^{N(H(\tau+t)) - N(H(\tau))} \\
&= \mathbb{E} [\mathbb{E}(u^{N(H(\tau+t)) - H(\tau)}) | H(\tau), H(\tau+t)]] \\
&= e^{-\int_\tau^{\tau+t} \lambda^{\alpha(s)} (1-u)^{\alpha(s)} ds}.
\end{aligned} \tag{3.3}$$

By a series expansion we have

$$\begin{aligned}
G(u, \tau, t) &= \sum_{r=0}^\infty \frac{(-1)^r}{r!} \left(\int_\tau^{\tau+t} \lambda^{\alpha(s)} (1-u)^{\alpha(s)} ds \right)^r \\
&= 1 + \sum_{r=1}^\infty \frac{(-1)^r}{r!} \int_{[\tau, \tau+t]^r} \lambda^{\beta_r(s)} (1-u)^{\beta_r(s)} ds_1 \dots ds_r \\
&= u^0 \left[1 + \sum_{r=1}^\infty \frac{(-1)^r}{r!} \int_{[\tau, \tau+t]^r} \lambda^{\beta_r(s)} ds_1 \dots ds_r \right] + \\
&\quad + \sum_{n=1}^\infty u^n \left[\sum_{r=1}^\infty \frac{(-1)^{n+r}}{r!} \int_{[\tau, \tau+t]^r} \lambda^{\beta_r(s)} \binom{\beta_r(s)}{n} ds_1 \dots ds_r \right].
\end{aligned}$$

Thus the increments of the SMPP have distribution

$$\Pr\{X(\tau+t) - X(\tau) = n\} = \begin{cases} \sum_{r=1}^\infty \frac{(-1)^{n+r}}{r!} \int_{[\tau, \tau+t]^r} \lambda^{\beta_r(s)} \binom{\beta_r(s)}{n} ds_1 \dots ds_r, & n \geq 1, \\ e^{-\int_\tau^{\tau+t} \lambda^{\alpha(s)} ds}, & n = 0, \end{cases} \tag{3.4}$$

Now we recall that additive processes are space-homogeneous (see [21], p.55), namely the transition probabilities are such that

$$\Pr\{X(t) \in B | X(s) = x\} = \Pr\{X(t) \in B - x | X(s) = 0\} = \Pr\{X(t) - X(s) \in B - x\},$$

for any $0 \leq s \leq t$ and any Borel set $B \subset \mathbb{R}$. Thus the desired result concerning the transition probabilities holds true. ■

Remark 3.2 By the same conditioning argument used in (3.3), we find that the Laplace transform of $X(t)$ reads

$$\mathbb{E}e^{-\eta N(H(t))} = e^{-\int_0^t \lambda^{\alpha(\tau)} (1-e^{-\eta})^{\alpha(\tau)} d\tau}$$

and then its Bernstein function is given by

$$f^*(\eta, t) = \lambda^{\alpha(t)} (1 - e^{-\eta})^{\alpha(t)}. \quad (3.5)$$

We can check that (3.5) can also be obtained by applying the definition involving the Lévy measure (3.1)

$$\begin{aligned} f^*(\eta, t) &= \int_0^\infty (1 - e^{-\eta x}) \nu_t^*(dx) \\ &= \lambda^{\alpha(t)} \sum_{k=1}^\infty \binom{\alpha(t)}{k} (-1)^{k+1} \int_0^\infty (1 - e^{-\eta x}) \delta_k(dx) \\ &= \lambda^{\alpha(t)} \sum_{k=1}^\infty \binom{\alpha(t)}{k} (-1)^{k+1} (1 - e^{-\eta k}) \\ &= \lambda^{\alpha(t)} (1 - e^{-\eta})^{\alpha(t)}. \end{aligned}$$

Remark 3.3 In the limiting case where the stability index is constant, namely $\alpha(s) = \alpha > 0$, the multistable subordinator H reduces to the classical stable subordinator and thus X is the classical space fractional process studied in [16], which is a time-homogeneous process. Indeed, it is straightforward to check that, if α is constant,

$$\beta_r(s) = \sum_{j=1}^r \alpha(s_j) = r\alpha$$

and, putting $\tau = 0$ by time homogeneity, expression (3.2) reduces to

$$\begin{aligned} p_n(t) &= \sum_{r=0}^\infty \frac{(-1)^{r+n}}{r!} \lambda^{\alpha r} t^r \binom{\alpha r}{n} \\ &= \sum_{r=0}^\infty \frac{(-1)^{r+n}}{r!} \lambda^{\alpha r} t^r \frac{\Gamma(\alpha r + 1)}{n! \Gamma(\alpha r - n + 1)}, \end{aligned}$$

which is the one-dimensional distribution computed in [16].

3.1 Governing equation

The Phillips' theorem states that, if $\{Y(t), t \geq 0\}$ is a Markov process with generator A and H is a subordinator with Bernstein function $f(\lambda)$, then $\{Y(H(t)), t \geq 0\}$ is a Markov process with generator $-f(-A)$ (for a deeper insight, consult [21] and [22]). This explains why the state probabilities $p_k(t) = P(X(t) = k)$ of the space-fractional Poisson process studied in [16] are governed by the following system of difference-differential equations

$$\begin{cases} \frac{d}{dt} p_k(t) = -\lambda^\alpha (I - B)^\alpha p_k(t) \\ p_k(0) = \delta_{k,0}, \end{cases} \quad (3.6)$$

where B is the shift operator such that $Bp_k(t) = p_{k-1}(t)$.

In [17], the Phillips' theorem has been partially extended to time-changed processes $Y(H(t))$, where H is a non-homogeneous subordinator with Bernstein function $f(u, t)$. Indeed, by means of a functional analysis approach, the authors proved that $\{Y(H(t)), t \geq 0\}$ is an additive process with time-dependent generator $-f(-A, t)$, at least when A is self-adjoint. Thus, it is not obvious that this fact also applies to the SMPP, since the generator A of a standard Poisson process is not self-adjoint. However, the following proposition confirms the Phillips' type form of the time-dependent generator.

Proposition 3.4 *The state probabilities of the SMPP solve the following system of difference-differential equations*

$$\begin{cases} \frac{d}{dt}p_k(t) = -\lambda^{\alpha(t)}(I - B)^{\alpha(t)}p_k(t), \\ p_k(0) = \delta_{k,0}. \end{cases} \quad (3.7)$$

Proof. Let us consider the distribution given in (3.4). Each multiple integral over $[0, t]^r$ is of order t^r , so that, for small time intervals, the distribution of the increments has the following form

$$\Pr\{X(t + dt) - X(t) = n\} = \begin{cases} 1 - \lambda^{\alpha(t)}dt + o(dt) & n = 0 \\ (-1)^{n+1}\lambda^{\alpha(t)}\binom{\alpha(t)}{n}dt + o(dt) & n \geq 1 \end{cases} \quad (3.8)$$

By using the expansion $(I - B)^{\alpha(t)} = \sum_{n=0}^{\infty} \binom{\alpha(t)}{n} (-1)^n B^n$, equation (3.7) can be written as

$$p_k(t + dt) = p_k(t)(1 - \lambda^{\alpha(t)}dt) + \sum_{n=1}^k p_{k-n}(t)\lambda^{\alpha(t)}(-1)^{n+1}\binom{\alpha(t)}{n}dt + o(dt)$$

which is the forward equation of an inhomogeneous Markov process whose infinitesimal (time-dependent) transition probabilities have just the form (3.8) and this concludes the proof. ■

3.2 Compound Poisson representation and jump times

The space-fractional Poisson process introduced in [16] is a counting process with upward jumps of arbitrary size. A fundamental property is that the waiting times between successive jumps, J_n , are i.i.d random variables with common distribution

$$\Pr\{J_n > \tau\} = e^{-\lambda^{\alpha}\tau}, \quad \forall n \geq 1.$$

Thus the jump times $T_n = J_1 + \dots + J_n$ follow a gamma distribution:

$$\Pr\{T_n \in dt\} = \frac{1}{\Gamma(n)}\lambda^{\alpha n}t^{n-1}e^{-\lambda^{\alpha}t}, \quad t \geq 0.$$

A difficulty arises in the SMPP case, where the waiting times J_n are neither independent nor identically distributed random variables and T_n cannot be obtained as the convolution of J_1, J_2, \dots, J_n . By using (3.2), the waiting time of the first jump has distribution

$$\Pr\{J_1 > t\} = \Pr\{X(t) = 0 | X(0) = 0\} = e^{-\int_0^t \lambda^{\alpha(s)} ds},$$

while the n^{th} waiting time is such that

$$\Pr\{J_n > t | J_1 + J_2 + \dots J_{n-1} = \tau\} = \Pr\{X(\tau + t) - X(\tau) = 0\} = e^{-\int_{\tau}^{\tau+t} \lambda^{\alpha(s)} ds}$$

and this shows that the variables $J_n, n \geq 1$, are stochastically dependent.

In order to find the distribution of T_n , it is convenient to note that the SMPP is an inhomogeneous CPP in the sense of [17]. In Section 1 we recalled that a non-homogeneous subordinator such that $\nu_t(\mathbb{R}^+) < \infty$, for each $t \geq 0$, reduces to a inhomogeneous CPP. As shown in [17], such a process can be constructed as

$$Y(t) = \sum_{j=1}^{P(t)} Y_j,$$

where $P(t)$ is a time-inhomogeneous Poisson process with intensity $g(t)$ and hitting times $T_j = \inf\{t \geq 0 : P(t) = j\}$, and Y_j are positive and non-stationary jumps, such that

$$\Pr\{Y_j \in dy | T_j = t\} = \psi(dy, t).$$

We recall that the Lévy measure of such a process has the form

$$\nu_t(dy) = g(t)\psi(dy, t), \quad (3.9)$$

whence $\nu_t(\mathbb{R}^+) = g(t) < \infty$, see [17] for details.

Theorem 3.5 *Let $X(t) = N(H(t))$ be the SMPP and consider the inhomogeneous CPP*

$$Y(t) = \sum_{j=1}^{P(t)} Y_j,$$

such that P is a inhomogeneous Poisson process with intensity $g(t) = \lambda^{\alpha(t)}$ and the Y_j have distribution

$$\psi(dx, t) = \Pr\{Y_j \in dx | T_j = t\} = \sum_{n=1}^{\infty} (-1)^{n+1} \binom{\alpha(t)}{n} \delta_n(dx).$$

Then

- i) X and Y are equal in the f.d.d.'s sense.*
- ii) The epochs T_j at which the jumps of X occur have marginal distributions*

$$\Pr\{T_j \in dt\} = \frac{(\int_0^t \lambda^{\alpha(s)} ds)^{j-1} e^{-\int_0^t \lambda^{\alpha(s)} ds}}{\Gamma(j)} \lambda^{\alpha(t)} dt. \quad (3.10)$$

Proof. *i)* X and Y are both inhomogeneous subordinators. Thus they are equal in the f.d.d.'s sense if and only if their Lévy measures coincide. By (3.9), the Lévy measure of Y is $\nu_t(dy) = g(t)\psi(dy, t)$ and it corresponds to (3.1).

ii) The process P has distribution

$$\Pr\{P(t) = k\} = e^{-\int_0^t \lambda^{\alpha(\tau)} d\tau} \frac{(\int_0^t \lambda^{\alpha(\tau)} d\tau)^k}{k!} \quad k \geq 0.$$

Now, P governs the epochs T_n , $n \geq 0$, at which the jumps of Y occur, i.e.

$$T_n = \inf\{t \geq 0 : P(t) = n\}.$$

To our aim, it is convenient to resort to the deterministic time change $t \rightarrow t'$ given by the transformation

$$t' = M(t) = \int_0^t \lambda^{\alpha(s)} ds, \quad (3.11)$$

where M is clearly a continuous and monotonic function. Thus $P(t)$ transforms into

$$\Pi(t') = P(M^{-1}(t')).$$

By virtue of the Mapping Theorem (see [7], p.18), $\Pi(t')$ is also a Poisson process. Moreover it is homogeneous with intensity 1 and its hitting times T'_j follow a $Gamma(1, j)$ distribution, i.e.

$$\Pr\{T'_j \in dt'\} = \frac{(t')^{j-1} e^{-t'}}{\Gamma(j)} dt'.$$

The hitting times T_j of $P(t)$ are the images of T'_j under the transformation M^{-1} : thus, by a simple transformation of the probability density of T'_j , we obtain (3.10). ■

3.3 Upcrossing times

Let \mathcal{T}_k be the time of the first upcrossing of the level k , i.e.

$$\mathcal{T}_k = \inf\{t \geq 0 : X(t) \geq k\}.$$

We now find two equivalent expressions for its distribution. The first one is a generalization of the result given in [18], p.8:

$$\begin{aligned} \Pr(\mathcal{T}_k > t) &= \Pr(X(t) < k) \\ &= \sum_{n=0}^{k-1} \Pr(X(t) = n) \\ &= \sum_{n=0}^{k-1} \int_0^\infty \Pr(N(s) = n) \Pr(H(t) \in ds) \\ &= \sum_{n=0}^{k-1} \int_0^\infty e^{-\lambda s} \frac{(\lambda s)^n}{n!} \Pr(H(t) \in ds) \\ &= \sum_{n=0}^{k-1} \frac{(-\lambda)^n}{n!} \frac{d^n}{d\lambda^n} \int_0^\infty e^{-\lambda s} \Pr(H(t) \in ds) \end{aligned}$$

$$= \sum_{n=0}^{k-1} \frac{(-\lambda)^n}{n!} \frac{d^n}{d\lambda^n} e^{-\int_0^t \lambda^{\alpha(\tau)} d\tau}. \quad (3.12)$$

The second one allows us to write the survival function of \mathcal{T}_k in terms of the state probability of the level k in the following way:

$$Pr(\mathcal{T}_k > t) = 1 - k \int_0^\lambda d\lambda' \frac{1}{\lambda'} Pr(X_{\lambda'}(t) = k). \quad (3.13)$$

We observe that, in both (3.12) and (3.13), we have that

$$Pr(\mathcal{T}_1 > t) = e^{-\int_0^t \lambda^{\alpha(\tau)} d\tau} = Pr(T_1 > t),$$

because the time when the first jump occurs (i.e. T_1) obviously coincides with the surpassing time of the level $k = 1$ (i.e. \mathcal{T}_1).

Here is the proof of (3.13), in the non-trivial case $k \geq 2$:

$$\begin{aligned} Pr(\mathcal{T}_k > t) &= Pr(X(t) < k) \\ &= Pr(X(t) = 0) + \sum_{n=1}^{k-1} Pr(X(t) = n) \\ &= e^{-\int_0^t \lambda^{\alpha(s)} ds} + \sum_{n=1}^{k-1} \sum_{r=1}^{\infty} \frac{(-1)^{n+r}}{r!} \int_{[0,t]^r} \lambda^{\beta_r(s)} \binom{\beta_r(s)}{n} ds_1 \dots ds_r \\ &= e^{-\int_0^t \lambda^{\alpha(s)} ds} + \sum_{r=1}^{\infty} \frac{(-1)^r}{r!} \int_{[0,t]^r} \lambda^{\beta_r(s)} \left(\sum_{n=1}^{k-1} (-1)^n \binom{\beta_r(s)}{n} \right) ds_1 \dots ds_r, \end{aligned} \quad (3.14)$$

where we used (3.4) putting $\tau = 0$. By using the following relation ¹

$$\sum_{n=0}^{k-1} (-1)^n \binom{x}{n} = (-1)^{k+1} \frac{k}{x} \binom{x}{k},$$

formula (3.14) reduces to

$$1 - k \sum_{r=1}^{\infty} \frac{(-1)^{r+k}}{r!} \int_{[0,t]^r} \frac{\lambda^{\beta_r(s)}}{\beta_r(s)} \binom{\beta_r(s)}{k} ds_1 \dots ds_r$$

and, by writing

$$\frac{\lambda^{\beta_r(s)}}{\beta_r(s)} = \int_0^\lambda (\lambda')^{\beta_r(s)-1} d\lambda',$$

equation (3.13) is immediately obtained.

¹Such a formula can be proved for $k = 1$ and then generalized to $k > 1$, by a standard use of the principle of induction.

4 Time-Multifractional Poisson Process

4.1 Inverse multistable process

Let H be a multistable subordinator. Since H is a cadlag process, with strictly increasing trajectories, and such that $H(0) = 0$ and $H(\infty) = \infty$ almost surely, then the hitting-time process

$$L(x) = \inf\{t \geq 0 : H(t) > x\} \quad (4.1)$$

is well defined and has continuous sample paths. Together with (4.1), the following definition holds

$$H(x^-) = \sup\{t \geq 0 : L(t) < x\}.$$

In the time-homogeneous case, it is well known that, if H is stable with index α , both H and its inverse L are self-similar, with exponent $1/\alpha$ and α respectively, that is the following relations hold in distribution (see, for example, [14]):

$$H(ct) = c^{\frac{1}{\alpha}} H(t), \quad L(ct) = c^\alpha L(t).$$

In the non-homogeneous case, the process H is not self-similar, but its local approximation has this property. More precisely, the multistable subordinator is localizable (see, for example, [9] and [17]), in the sense that

$$\lim_{r \rightarrow 0} \frac{H(t + rT) - H(t)}{r^{\frac{1}{\alpha(t)}}} \stackrel{\text{law}}{=} Z_t(T), \quad (4.2)$$

where Z_t is the local (or tangent) process at t and consists of a homogeneous stable process with index $\alpha(t)$. We now investigate the behaviour of the inverse process.

Proposition 4.1 *The process L defined in (4.1) is localizable, and the tangent process is given by the inverse \mathcal{L}_t of Z_t .*

Proof. By (4.2), for each $t \geq 0$, we can write

$$\lim_{r \rightarrow 0} \Pr\left(\frac{H(t + rT) - H(t)}{r^{\frac{1}{\alpha(t)}}} \leq w\right) = \Pr(Z_t(T) \leq w) = \Pr\{\mathcal{L}_t(w) \geq T\}. \quad (4.3)$$

Since H has independent increments, $H(t + rT) - H(t)$ is independent of $H(t)$. So we can condition on $H(t) = x$, without changing the left-hand side of (4.3), which can be written as

$$\begin{aligned} & \lim_{r \rightarrow 0} \Pr\left(H(t + rT) - H(t) \leq wr^{\frac{1}{\alpha(t)}} \mid H(t) = x\right) \\ &= \lim_{r \rightarrow 0} \Pr\left(L(x + wr^{\frac{1}{\alpha(t)}}) - L(x) \geq rT\right) \\ &= \lim_{r' \rightarrow 0} \Pr\left(\frac{L(x + wr') - L(x)}{r'^{\alpha(t)}} \geq T\right), \end{aligned} \quad (4.4)$$

where, in the last step, we made the substitution $r' = r^{\frac{1}{\alpha(t)}}$. Thus

$$\lim_{r' \rightarrow 0} \Pr\left(\frac{L(x + wr') - L(x)}{r'^{\alpha(t)}} \geq T\right) = \Pr\{\mathcal{L}_t(w) \geq T\}$$

and the proof is complete. ■

4.2 Paths and distributional properties

Let N be a Poisson process with intensity $\lambda > 0$, and let L be the inverse of a multi-stable subordinator independent of N . We define the TMPP as the time-changed process $N(L(t))$. Since N is one-stepped and L is continuous, then $N(L(t))$ is also a one-stepped continuous time random walk defined as

$$N(L(t)) = n \iff T_n < t < T_{n+1} \quad n = 0, 1, 2, \dots$$

where $T_0 = 0$ a.s. and, for $n \geq 1$, $T_n = J_1 + \dots + J_n$, J_n being the waiting time for the state n . The construction of the process is contained in the following result.

Theorem 4.2 *The time-changed Poisson process $\{N(L(t)), t \geq 0\}$ is a one-stepped counting process with independent waiting times $J_n, n \geq 1$, each having Laplace transform*

$$\mathbb{E}e^{-\eta J_n} = \int \int_{0 < u < v < \infty} \frac{\lambda^n e^{-\lambda v} u^{n-2}}{\Gamma(n-1)} e^{-\int_u^v \eta^{\alpha(\tau)} d\tau} du dv \quad n \geq 2 \quad (4.5)$$

$$\mathbb{E}e^{-\eta J_1} = \int_0^\infty dw \lambda e^{-\lambda w} e^{-\int_0^w \eta^{\alpha(\tau)} d\tau}. \quad (4.6)$$

Proof. Let $W_n, n \geq 1$ be the i.i.d waiting times between jumps of a Poisson process, so that $\Pr(W_n \in dw) = \lambda e^{-\lambda w} dw$. Let $V_n = W_1 + W_2 + \dots + W_n, n \geq 1$, be the hitting times of $N(t)$, each having distribution $\Pr(V_n \in du) = \Gamma(n)^{-1} \lambda^n e^{-\lambda u} u^{n-1} du, u > 0$.

The joint distribution of two successive hitting times reads

$$\begin{aligned} \Pr(V_{n-1} \in du, V_n \in dv) &= \Pr(V_{n-1} \in du, W_n \in d(v-u)) \\ &= \Pr(V_{n-1} \in du) \Pr(W_n \in d(v-u)) \\ &= \frac{\lambda^n e^{-\lambda v} u^{n-2}}{\Gamma(n-1)} du dv \quad 0 < u < v < \infty. \end{aligned}$$

Now let $T_1 \dots T_n$ be the hitting times of $N(L(t))$, such that

$$T_n = \sup\{t \geq 0 : L(t) < V_n\}.$$

Since L is the right continuous inverse of H , it follows that $T_n = H(V_n^-)$ and this, together with the fact that H is a.s. continuous for any $t \geq 0$ (see [17]), implies that $T_n = H(V_n)$ in distribution. The waiting times between jumps of $N(L(t))$ are defined as $J_n = T_n - T_{n-1}$, where $n \geq 1$. For $n = 1$ we have that

$$\mathbb{E}e^{-\eta J_1} = \mathbb{E}e^{-\eta H(W_1)} = \mathbb{E}[\mathbb{E}(e^{-\eta H(W_1)} | W_1)] = \int_0^\infty dw \lambda e^{-\lambda w} e^{-\int_0^w \eta^{\alpha(\tau)} d\tau}$$

while, for $n \geq 2$

$$\begin{aligned} \mathbb{E}e^{-\eta J_n} &= \mathbb{E}e^{-\eta(T_n - T_{n-1})} \\ &= \mathbb{E}e^{-\eta(H(V_n) - H(V_{n-1}))} \\ &= \mathbb{E}[\mathbb{E}(e^{-\eta(H(V_n) - H(V_{n-1}))} | V_{n-1}, V_n)] \end{aligned}$$

$$\begin{aligned}
&= \int \int_{0 < u < v < \infty} \mathbb{E} e^{-\eta(H(v)-H(u))} P(V_n \in dv, V_{n-1} \in du) \\
&= \int \int_{0 < u < v < \infty} e^{-\int_u^v \eta^{\alpha(\tau)} d\tau} \frac{\lambda^n e^{-\lambda v} u^{n-2}}{\Gamma(n-1)} du dv,
\end{aligned}$$

and the proof is complete. ■

Remark 4.3 In the time-homogeneous case, where $\alpha(s) = \alpha \in (0, 1)$, the random time L reduces to the inverse stable subordinator with index α . A slight calculation shows that expressions (4.5) and (4.6) become independent of the state n and have the following form

$$\mathbb{E} e^{-\eta J_n} = \frac{\lambda}{\eta^\alpha + \lambda}, \quad \forall n \geq 1.$$

Thus we have a renewal process with i.i.d waiting times which satisfy

$$\Pr(J_n > t) = E_\alpha(-\lambda t^\alpha),$$

where

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(1 + \alpha k)}$$

is the Mittag-Leffler function. So, in the homogeneous case, $N(L(t))$ reduces to the celebrated time-fractional Poisson process, which is a renewal process with Mittag-Leffler waiting times (see, for example, [3]). In such a case, the one-dimensional state probabilities $p_k(t) = \Pr(N(L(t)) = k)$ solve the following system of fractional difference-differential equations

$$\begin{cases} \frac{\partial^\alpha}{\partial t^\alpha} p_k(t) = -\lambda p_k(t) + \lambda p_{k-1}(t) & k \geq 1 \\ \frac{\partial^\alpha}{\partial t^\alpha} p_0(t) = -\lambda p_0(t) \\ p_k(0) = \delta_{k,0}, \end{cases} \quad (4.7)$$

where

$$\frac{\partial^\alpha}{\partial t^\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} f'(s) ds$$

is the Caputo derivative of order $\alpha \in (0, 1)$. As shown in [4], the solution to (4.7) is such that

$$\tilde{p}_k(s) = \int_0^\infty e^{-st} p_k(t) dt = \frac{\lambda^k s^{\alpha-1}}{(s^\alpha + \lambda)^{k+1}} \quad k \geq 0 \quad (4.8)$$

We now find the multifractional analogue of formula (4.8).

Proposition 4.4 Let $N(L(t))$ be a TMPP and let $p_k(t) = \Pr\{N(L(t)) = k\}$ be its state probabilities. Then

$$\tilde{p}_0(s) = \frac{1}{s} - \frac{1}{s} \int_0^\infty dw \lambda e^{-\lambda w} e^{-\int_0^w s^{\alpha(\tau)} d\tau}, \quad (4.9)$$

$$\tilde{p}_k(s) = \int_0^\infty dx e^{-\int_0^x s^{\alpha(\tau)} d\tau} \frac{\lambda^k x^k e^{-\lambda x} (kx^{-1} - \lambda)}{s\Gamma(k+1)}, \quad k \geq 1. \quad (4.10)$$

Proof. Consider that $p_0(t) = P(J_1 > t)$. By deriving this relation with respect to t and taking the Laplace transform, (4.6) leads to (4.9).

Let now $T_k, k \geq 1$ be the hitting times of $N(L(t))$. As explained in the proof of the previous theorem, $T_k = H(V_k)$ in distribution, that is

$$\mathbb{E}e^{-sT_k} = \int_0^\infty dx \frac{\lambda^k x^{k-1} e^{-\lambda x}}{\Gamma(k)} e^{-\int_0^x s^\alpha(\tau) d\tau}. \quad (4.11)$$

By considering that

$$\Pr\{N(L(t)) \geq k\} = \Pr\{T_k < t\}, \quad k \geq 1,$$

we have that $\Pr\{N(L(t)) = k\} = \Pr\{T_k < t\} - \Pr\{T_{k+1} < t\}$. By deriving with respect to t and taking the Laplace transform, we obtain

$$s\tilde{p}_k(s) = \mathbb{E}e^{-sT_k} - \mathbb{E}e^{-sT_{k+1}}$$

and, using (4.11), formula (4.10) follows. ■

It is straightforward to note that (4.9) and (4.10) reduce to (4.8) by assuming $\alpha(t)$ to be constant with respect to t .

4.3 Concluding remarks

Unfortunately, in the time-inhomogeneous case, the connection with fractional calculus is not immediate as in the classical case. Indeed, in [17] the authors proposed an equation governing Markovian processes time-changed via the inverses of inhomogeneous subordinators. Such equation involves generalized fractional derivatives, but it is not easy to handle, especially because it does not involve the distribution of the time-changed process only, but also the distributions of both the original Markov process and the operational time.

We finally observe that our construction of the TMPP extends to the inverses of arbitrary non-homogeneous subordinators provided that $\nu_t(\mathbb{R}^+) = \infty$. Indeed, let N be an ordinary Poisson process and let L be the inverse of any non-homogeneous subordinator. Then $N(L(t))$ is a counting process with independent intertimes given by

$$\mathbb{E}e^{-\eta J_n} = \int \int_{0 < u < v < \infty} \frac{\lambda^n e^{-\lambda v} u^{n-2}}{\Gamma(n-1)} e^{-\int_u^v f(\eta, \tau) d\tau} du dv \quad n \geq 2 \quad (4.12)$$

$$\mathbb{E}e^{-\eta J_1} = \int_0^\infty dw \lambda e^{-\lambda w} e^{-\int_0^w f(\eta, \tau) d\tau} \quad (4.13)$$

To prove this, it is sufficient to adapt the same construction given in the proof of Theorem 4.2, using the inverse process of an inhomogeneous subordinator with Bernstein function of the form $f(x, t)$. Of course, in the homogeneous case, where f is independent of t , we obtain a renewal process with i.i.d intertimes such that

$$\mathbb{E}e^{-\eta J_n} = \frac{\lambda}{\lambda + f(s)}, \quad n \geq 1,$$

which has been analysed in [13].

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